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## NUMERICAL METHOD FOR SOLVING A LINEAR BOUNDARY VALUE PROBLEM FOR FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS

**Abstract.** Numerical method to solve the linear boundary value problem for the Fredholm integro-differential equations with degenerate kernel is proposed. Dividing interval into  $N$  parts and introducing additional parameters as the values of solution at the left-end points of subintervals origin problem is reduced to the multipoint boundary value problem for the system of integro-differential equations with parameters. At the fixed values of parameters the special Cauchy problems for the system of integro-differential equations are solved. Introducing the additional parameters allows as the solvability of the boundary value problem to reduce to the solvability of system of linear algebraic equation with respect to introduced parameters. The Cauchy problems for the ordinary differential equations and evaluating the definite integrals on subintervals are the main auxiliary problems of method proposed. The Cauchy problems are solved by the Bulirsch-Stoer method and definite integrals are determined by the Simpson method.

**Keywords:** Fredholm integro-differential equations, linear boundary value problem, parametrization, Bulirsch-Stoer method.

Various problems of physics, engineering, biology, etc. lead to the study of integro-differential equations and to the formulation of related specific tasks. In connection with this, the theory of such equations has attracted the attention of mathematicians. Qualitative properties of problems for the Fredholm integro-differential equations and methods for solving these problems are considered in the works of many others [1-11].

In the present paper, we consider the linear boundary value problem for Fredholm integro-differential equation:

$$\frac{dx}{dt} = A(t)x + \sum_{k=1}^m \int_0^T \varphi_k(t)\psi_k(s)x(s)ds + f(t), t \in (0, T), x \in R^n \quad (1)$$

$$Bx(0) + Cx(T) = d, d \in R^n \quad (2)$$

where the matrices  $A(t)$ ,  $\varphi_k(t)$ ,  $\psi_k(s)$ ,  $0 < k < m$  and vector  $f(t)$  are continuous on  $[0, T]$ . A solution to problem (1) and (2) is a vector function  $x(t)$ , continuous on  $[0, T]$  and continuously differentiable on  $(0, T)$ , satisfying the integro-differential equation (1) and boundary condition (2).

Given the points:  $t_0 = 0 < t_1 < \dots < t_N = T$ , and let  $\Delta_N$  denote the partition of interval  $[0, T]$  into  $N$  subintervals  $[0, T] = \bigcup_{r=1}^N [t_{r-1}, t_r]$ . The case, when the interval  $[0, T]$  is not divided into parts, we denote by  $\Delta_1$ .

Let  $x_r(t)$  be the restriction of function  $x(t)$  to the  $r$ -th interval  $[t_{r-1}, t_r]$ , i.e.  $x_r(t) = x(t)$ , for  $t \in [t_{r-1}, t_r]$ ,  $r = 1, N$ .

Introducing the additional parameters  $\lambda_r = x_r(t_{r-1})$  and performing a replacement of the function  $u_r(t) = x_r(t) - \lambda_r$  on each  $r$ -th interval, we obtain the following boundary value problem with parameters:

$$\frac{du_r}{dt} = A(t)(u_r + \lambda_r) + \sum_{j=1}^N \sum_{k=1}^m \int_{t_{j-1}}^{t_j} \varphi_k(t)\psi_k(s)(u_r(s) + \lambda_r)ds + f(t) \quad (3)$$

$$t \in (t_{r-1}, t_r), \quad r = \overline{1, N},$$

$$u_r(t_{r-1}) = 0, \quad r = 1 \dots N \quad (4)$$

$$B\lambda_1 + C\lambda_N + C \lim_{t \rightarrow T-0} u_N(t) = d \quad (5)$$

$$\lambda_p + \lim_{t \rightarrow t_p-0} u_p(t) - \lambda_{p+1} = 0, \quad p = 1 \dots N - 1 \quad (6)$$

where (6) are conditions for matching the solution at the interior points of the partition  $\Delta_N$ . Note, that conditions (6) and integro-differential equations (3) also ensure the continuity of solution's derivatives at these points.

Using the fundamental matrix  $X_r(t)$  of differential equation  $dx/dt = A(t)x$  on  $[t_{r-1}, t_r]$ , we reduce the special Cauchy problem for the system of integro-differential equations with parameters (2.3), (2.4) to the equivalent system of integral equations.

$$\begin{aligned} u_r(t) = & X_r \int_{t_{r-1}}^t X_r^{-1}(\tau)A(\tau)d\tau \lambda_r + \\ & + X_r \int_{t_{r-1}}^t X_r^{-1}(\tau)A(\tau) \sum_{j=1}^N \sum_{k=1}^m \int_{t_{j-1}}^{t_j} \varphi_k(\tau)\psi_k(s)(u(s)_r + \lambda_r)dsd\tau + \\ & + X_r \int_{t_{r-1}}^t X_r^{-1}(\tau)A(\tau)f(\tau)d\tau, \quad t \in [t_{r-1}, t_r], \quad r = 1 \dots N \end{aligned} \quad (7)$$

Introduce the notation

$$\mu_k = \sum_{j=1}^N \int_{t_{j-1}}^{t_j} \psi_k(s)u_j(s)ds$$

Multiplying both sides of (7) by  $\psi_p(t)$ , integrating on the interval  $[t_{r-1}, t_r]$  and summing up over  $r$ , we have the system of linear algebraic equations with respect to  $\mu = (\mu_1, \dots, \mu_m) \in R^{nm}$ :

$$\mu_p = \sum_{k=1}^m G_{p,k}(\Delta_N)\mu_k + \sum_{r=1}^N V_{p,r}(\Delta_N)\lambda_r + g_p(f, \Delta_N), \quad p = 1 \dots m \quad (8)$$

with the  $(n \times n)$  matrices

$$G_{p,k}(\Delta_N) = \sum_{r=1}^N \int_{t_{r-1}}^{t_r} \psi_r(\tau)X_r(\tau) \int_{t_{r-1}}^{\tau} X_r^{-1}(\tau)\phi_k(s)dsd\tau \quad (9)$$

$$V_{p,r}(\Delta_N) = \int_{t_{r-1}}^{t_r} \psi_p(\tau) X_r(\tau) \int_{t_{r-1}}^{\tau} X_r^{-1}(\tau) A(s) ds d\tau +$$

$$+ \sum_{j=1}^N \sum_{k=1}^m \int_{t_{j-1}}^{t_j} \psi_p(\tau) X_j(\tau) \int_{t_{j-1}}^{\tau} X_j^{-1}(\tau) \phi_k(\tau) d\tau d\tau \int_{t_{r-1}}^{t_r} \psi_k(s) ds$$
(10)

and vectors of dimension  $n$

$$g_p(f, \Delta_N) = \sum_{r=1}^N \int_{t_{r-1}}^{t_r} \psi_p(\tau) X_r(\tau) \int_{t_{r-1}}^{\tau} X_r^{-1}(\tau) f(s) ds d\tau$$
(11)

Using these matrices we can rewrite the system (8) in the form

$$[I - G(\Delta_N)]\mu = V(\Delta_N)\lambda + g(f, \Delta_N),$$
(12)

where  $I$  is the identity matrix of dimension  $nm$ .

Essential requirement to the partition is its regularity. Partition  $\Delta_N$  is called regular if the matrix  $I - G(\Delta_N)$  is invertible. It is established that the invertibility of system's matrix is equivalent to the well-posedness of considered boundary value problem [2]. Assume the matrix  $I - G(\Delta_N)$  is invertible and  $[I - G(\Delta_N)]^{-1} = (M_{k,p}(\Delta_N))$ . Then according to (12) the elements of the vector  $\mu \in R^{nm}$  are determined by the equalities

$$\mu_k = \sum_{j=1}^N \left( \sum_{p=1}^m M_{k,p}(\Delta_N) V_{p,j}(\Delta_N) \right) \lambda_j + \sum_{p=1}^m M_{k,p}(\Delta_N) g_p(f, \Delta_N),$$
(13)

Substituting the right-hand side of (13) instead of  $\mu_k$ , we get the representation of functions  $u_r(t)$  via  $\lambda_j$

$$u_r(t) = \sum_{j=1}^N \left\{ \sum_{k=1}^m X_r(t) \int_{t_{r-1}}^t X_r^{-1}(\tau) \phi_k(\tau) \right.$$

$$\times \left[ \sum_{p=1}^m M_{k,p}(\Delta_N) V_{p,j}(\Delta_N) + \int_{t_{j-1}}^{t_j} \psi_k(s) ds \right] \lambda_j +$$

$$+ X_r(t) \int_{t_{r-1}}^t X_r^{-1}(\tau) A(\tau) d\tau \lambda_r + X_r(t) \int_{t_{r-1}}^t X_r^{-1} \times$$

$$\times \left[ \sum_{k=1}^m \phi_k(\tau) \sum_{p=1}^m M_{k,p}(\Delta_N) g_p(f, \Delta_N) + f(\tau) \right] d\tau,$$
(14)

Introduce notation

$$D_{r,j}(\Delta_N) = \sum_{h=1}^{2n} X_r(t) \int_{t_{r-1}}^t X_r^{-1}(\tau) \phi_h(\tau) \times \\ \times \left[ \sum_{p=1}^{2n} M_{h,p}(\Delta_N) V_{p,j}(\Delta_N) + \int_{t_{j-1}}^{t_j} \psi_h(s) ds \right] \tag{15}$$

$$D_{r,p}(\Delta) = \sum_{h=1}^{2n} X_r(t_r) \int_{t_{r-1}}^{t_r} X_r^{-1}(\tau) \phi_h(\tau) d\tau \times \\ \times \left[ \sum_{p=1}^{2n} M_{h,p}(\Delta_N) V_{p,j}(\Delta_N) + \int_{t_{j-1}}^{t_j} \psi_h(s) ds \right] + \tag{16}$$

$$+ X_r(t_r) \int_{t_{r-1}}^{t_r} X_r^{-1}(\tau) A(\tau) d\tau \\ F_r(\Delta_N) = \sum_{h=1}^{2n} X_r(t_r) \int_{t_{r-1}}^{t_r} X_r^{-1}(\tau) A(\tau) d\tau \sum_{p=1}^{2n} M_{h,p}(\Delta_N) g_p(f, \Delta_N) + \\ + \sum_{h=1}^{2n} X_r(t_r) \int_{t_{r-1}}^{t_r} X_r^{-1}(\tau) f(\tau) d\tau, r = 1 \dots N, \tag{17}$$

Then from (14) we have

$$\lim_{t \rightarrow t_r - 0} u_r(t) = \sum_{j=1}^N D_{r,j}(\Delta_N) \lambda_j + F_r(\Delta_N). \tag{18}$$

Substituting the right-hand side of (18) into the boundary condition (5) and conditions of matching solution (6), we obtain the following system of linear algebraic equations with respect to parameters  $\lambda_r, r = (1, N)$ :

$$[B + CD_{N,1}(\Delta_N)] \lambda_1 + \sum_{j=2}^{N-1} CD_{N,j}(\Delta_N) \lambda_j + \\ + C[I + D_{N,N}(\Delta_N)] \lambda_N = d - CF_N(\Delta_N), \tag{19}$$

$$[I + D_{p,p}(\Delta_N)] \lambda_p - [I + D_{p,p+1}(\Delta_N)] \lambda_{p+1} + \\ + \sum_{j=1}^N D_{p,j}(\Delta_N) \lambda_j = -F_p(\Delta_N), p = 1 \dots N - 1 \tag{20}$$

By denoting the matrix corresponding to the left-hand side of the system of equations (19), (20) by  $Q$ , the system can be written as the following:

$$Q_*(\Delta_N) \lambda = -F_*(\Delta_N), \lambda \in R^{nN}, \tag{21}$$

where

$$F_*(\Delta_N) = \{-d + CF_N(\Delta_N), F_1(\Delta_N), \dots, F_{N-1}(\Delta_N)\} \in R^{nN} \tag{22}$$



Solving equation (21) we find  $\lambda$  and substitute it to (14) to calculate  $u$ . Finally, performing a replacement of the function  $u_r(t) = x_r(t) - \lambda_r$  on each  $r$ -th interval, we obtain the values of the vector function  $x(t)$ .

Consider the Cauchy problems for ordinary differential equations on subintervals.

$$\frac{dx}{dt} = A(t)x + P(t), x(t_{r-1}) = 0, t \in [t_{r-1}, t_r], r = 1 \dots N \quad (23)$$

Here  $P(t)$  is square matrix or vector of dimension  $n$ , continuous on  $[0, T]$ .

Let's denote the solution to the Cauchy problem (23) by  $E_{*,r} = (A(\cdot), P(\cdot), t)$ .

Solving the Cauchy problem for the ordinary differential equations we obtain  $E_{*,r} = (A(\cdot), P(\cdot), t)$  and then evaluate the integrals

$$\begin{aligned} \hat{\psi}_{p,r} &= \int_{t_{r-1}}^{t_r} \psi(t) dt, \hat{\psi}_{p,r}(A) = \int_{t_{r-1}}^{t_r} \psi(t) E_{*,r}(A(\cdot), A(\cdot), t) dt \\ \hat{\psi}_{p,r}(f) &= \int_{t_{r-1}}^{t_r} \psi(t) E_{*,r}(A(\cdot), f(\cdot), t) dt, \hat{\psi}_{p,r}(\phi) = \int_{t_{r-1}}^{t_r} \psi(t) E_{*,r}(A(\cdot), \phi(\cdot), t) dt \end{aligned} \quad (24)$$

From the equalities (24) we calculate matrices  $G, V, g$ . Consequently we obtain the matrices  $Q, F$  and form the system of linear algebraic equations with respect to parameter  $\lambda$ . Solving composed equation we find the parameter  $\lambda$ . The Cauchy problems for ordinary differential equations for the found values of parameter we solve using Bulirsch–Stoer method [12]. Bulirsch–Stoer algorithm is a method for the numerical solution of ordinary differential equations which combines three powerful ideas: Richardson extrapolation, the use of rational function extrapolation in Richardson-type applications, and the modified midpoint method, to obtain numerical solutions to ordinary differential equations (ODEs) with high accuracy and comparatively little computational effort.

Now we consider on  $[0, T]$  the linear boundary value problem for Fredholm integro-differential equation (1) and (2) with degenerate kernel, where

$$\begin{aligned} T = 1, m = 2, A(t) &= \begin{pmatrix} 0 & t \\ t^2 & 0 \end{pmatrix}, B = -C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, d = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ f(t) &= \begin{pmatrix} 2\pi \cos(2\pi t), -t \cos(2\pi t) + (t^4 \exp(t/4)) / (4\pi) \\ -2\pi \sin(2\pi t) - t^2 \sin(2\pi t) \end{pmatrix} \\ \phi_1(t) &= \begin{pmatrix} t^3 \exp(t/4) & t \\ t & 2t^2 \exp(t/4) \end{pmatrix}, \phi_2(t) = \begin{pmatrix} \frac{t^4}{2} \exp(t/4) & 0 \\ 0 & t^4 \exp(t/4) \end{pmatrix} \\ \psi_1(t) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \psi_2(t) = \begin{pmatrix} t - \frac{1}{2} & 0 \\ 0 & t - \frac{1}{2} \end{pmatrix}, n = 2 \end{aligned}$$

Here the matrix of differential part is variable and the construction of fundamental matrix breaks down. We use the numerical implementation of algorithm. For the integro-differential equation in the given problem any partition of interval  $[0, T]$ , including  $\Delta_1$ , is regular. Accuracy of solution depends on the accuracy of solving the Cauchy problem on subintervals and evaluating of definite integrals. Numerical solution for Cauchy problem and evaluation of definite integrals are implemented using Bulirsch–Stoer method and Simpson's method, respectively. We provide the results of the numerical implementation of algorithm by partitioning the interval  $[0, 1]$  with step  $h = 0.5$  and partitioning the subintervals  $[0, 0.5]$  and  $[0.5, 1]$  with step  $h_1 = h_2 = 0.05$ .

| t      | $x_1^*(t)$ | $x_2^*(t)$ | t      | $x_1^*(t)$ | $x_2^*(t)$ |
|--------|------------|------------|--------|------------|------------|
| 0      | -0.0000    | 0.0000     | 0.5000 | 1.0000     | -1.0000    |
| 0.0500 | 0.3090     | -0.3090    | 0.5500 | 0.9511     | -0.9511    |
| 0.1000 | 0.5878     | -0.5878    | 0.6000 | 0.8090     | -0.8090    |
| 0.1500 | 0.8090     | -0.8090    | 0.6500 | 0.5878     | -0.5878    |
| 0.2000 | 0.9511     | -0.9511    | 0.7000 | 0.3090     | -0.3090    |
| 0.2500 | 1.0000     | -1.0000    | 0.7500 | 0.0000     | 0.0000     |
| 0.3000 | 0.9511     | -0.9511    | 0.8000 | -0.3090    | 0.3090     |
| 0.3500 | 0.8090     | -0.8090    | 0.8500 | -0.5878    | 0.5878     |
| 0.4000 | 0.5878     | -0.5878    | 0.9000 | -0.8090    | 0.8090     |
| 0.4500 | 0.3090     | -0.3090    | 0.9500 | -0.9511    | 0.9511     |
| 0.5000 | -0.0000    | -0.0000    | 1.0000 | -1.0000    | 1.0000     |

Here is  $x_1^*(t)$  and  $x_2^*(t)$  are approximate solutions of the integro-differential equation. Exact solution to the given problem is

$$x^*(t) = \begin{pmatrix} \sin(2\pi t) \\ \cos(2\pi t) \end{pmatrix}$$

and the following estimate is true

$$\max_{j=1..20} \|x(t_j) - x^*(t_j)\| < 0.000003552$$

whereas the results obtained from the Runge-Kutta 4<sup>th</sup> order method is:

$$\max_{j=1..20} \|x(t_j) - x^*(t_j)\| < 0.000003662$$

Moreover, Bulirsch-Stoer method showed better performance in terms of computation time: spent\_time (Bulirsch-Stoer) = 3.1511s versus spent\_time (Runge-Kutta 4) = 3.9654s, which is 25.8% faster.

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### ФРЕДГОЛЬМ ИНТЕГРО-ДИФФЕРЕНЦИАЛДЫҚ ТЕНДЕУІ ҮШІН СЫЗЫҚТЫҚ ШЕТТІК ЕСЕПТІ ШЕШУДІҢ САНДЫҚ ӘДІСІ

**Аннотация.** Өзегі айныған Фредгольм интегро-дифференциалдық теңдеуі үшін сызықтық шеттік есепті шешудің сандық әдісі ұсынылды. Аралықты  $N$  бөлікке бөлу және қосымша параметрлерді шешімнің бөлім-шенің сол жақ нүктелеріндегі мәні ретінде еңгізу бастапқы есепті параметрлері бар интегро-дифференциалдық теңдеулер жүйесі үшін көпнүктелі шеттік есепке алып келеді. Параметрлердің бекітілген мәндерінде интегро-дифференциалдық теңдеулер жүйесі үшін арнайы Коши есебі шешіледі. Қосымша параметрлерді еңгізу шеттік есептің шешілімділігін еңгізілген параметрлерге байланысты сызықтық алгебралық теңдеулер жүйесінің шешілімділігіне келтіреді.

Бөлімшелердегі жәй дифференциалдық теңдеулер үшін Коши есебі және анықталған интегралдарды есептеу ұсынылған әдістің негізгі қосалқы есептері болып табылады. Коши мәселелері Булирш –Штер әдісімен шешіледі және анықталған интегралдар Симпсон әдісімен есептеледі.

**Түйін сөздер:** Фредгольм интегро-дифференциалдық теңдеуі, сызықтық шеттік есеп, параметрлеу, Булирш-Штер әдісі.

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### ЧИСЛЕННЫЙ МЕТОД РЕШЕНИЯ ЛИНЕЙНОЙ КРАЕВОЙ ЗАДАЧИ ДЛЯ ИНТЕГРО-ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ ФРЕДГОЛЬМА

**Аннотация.** Предложен численный метод решения линейной краевой задачи для интегро-дифференциальных уравнений Фредгольма с вырожденным ядром. Разбиение интервала на  $N$  частей и введение дополнительных параметров в качестве значений решения в левых точках подинтервалов исходную задачу сводит к многоточечной краевой задаче для системы интегро-дифференциальных уравнений с параметрами. При фиксированных значениях параметров решаются специальные задачи Коши для системы интегро-дифференциальных уравнений. Введение дополнительных параметров позволяет нам разрешимость краевой задачи сводить к разрешимости системы линейных алгебраических уравнений относительно введенных параметров. Задачи Коши для обыкновенных дифференциальных уравнений и вычисление определенных интегралов на подинтервалах являются основными вспомогательными задачами предложенного метода. Задачи Коши решаются методом Булирша-Штера, а определенные интегралы вычисляются методом Симпсона.

**Ключевые слова:** интегро-дифференциальное уравнение Фредгольма, линейная краевая задача, параметризация, метод Булирша –Штера.

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